Българска академия на науките • Bulgarian Academy of Sciences
Аерокосмически изследвания в България • 13 • Aerospace Research in Bulgaria
София • 1997 • Sofia

On the perturbative gravity and quantum gravity theory on a curved background. I. Second-order gravitational Lagrangian decomposition*

(Revised version)

Bogdan Dimitrov

High Energy Astrophysics Section, Space Research Institute, Bulgarian Academy of Sciences

Introduction and motivation for this research

The theory of gravitons-quantized gravitational perturbations, has important implications in the present wormhole theory [1] and in the contemporary inflationary universe scenario [2].

The old, fundamental theory of quantum gravity, developed by Bryce Dewitt [3-6], provides the general theoretical background for the definition of the S-matrix, the effective action and etc. - all of them relative to a background field. This field was not specified, but most important, it was assumed to be fixed and asymptotically flat. As Bryce Dewitt himself remarks in his well-known paper [3], "the extension of the theory to worlds in which space-time is not asymptotically flat remains a program for the future'. Later on, the necessity to construct a gauge invariant effective action, based on the loop expansion [7] and the inclusion of ficticious particle loops [8, 9] lead to an investigation of the

*Work supported in part by the National Science Foundation of Bulgaria under grant № NIP-753

perturbative gravitational Lagrangian structure. In this Lagrangian the metric is decomposed into a flat background metric, denoted by $\delta^{\mu\nu}$ (δ the n-dimensional Kroneker symbol) and also a fluctuating metric $h_{\mu\nu}$, identified with the graviton field [9]:

$$g^{\mu\nu} \equiv \delta^{\mu\nu} + h^{\mu\nu}. \tag{1}$$

Again, the assumption about a flat background metric is a very convenient one because from the invariance of the metric under an infinitesimal gauge transformation:

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \tag{2}$$

one can compute also the induced change in the graviton field $h^{\mu\nu}$ [9].

Further, the necessity to investigate the one-loop divergencies of the scalar field when the gravitational field is treated as an external field [10], provided the motivation for decomposing the gravitational Lagrangian to second order on an arbitrary (non-flat) background. However, the decomposition turned out to be rather cumbersome and inconvenient to deal with [10, 11].

In a more concise and compact manner, the perturbed to second order Lagrangian was derived in [12] by using covariant derivatives in respect to the background metric. As is well known [13], the notion of a covariant derivative implies an important physical meaning, related to the change of a tensor field under an infinitesimal parallel transport along a given contour. It is instructive to write down only the gravitational part of the quadratic Lagrangian (for Einstein-Maxwell fields), derived in [12]:

$$\int d^4x(-g)^{\frac{1}{2}}\Big[-\tfrac{1}{2}\left(D_\nu h_{\alpha\beta}\right)P^{\alpha\beta\rho\sigma}\left(D^\nu h_{\rho\sigma}\right)+\tfrac{1}{2}\left(h_\mu-\tfrac{1}{2}D_\nu h\right)^2 + \tfrac{1}{2}h_{\alpha\beta}\left(X_g+X_e\right)^{\alpha\beta\rho\sigma}\Big] \tag{3}$$

where the symbot D denotes a covariant derivative and $P^{\alpha\beta\rho\sigma}$, X_e and X_g are fourth-rank tensors, composed only of background fields. In spite of the use of an arbitrary background metric, an assumption was made in [12], that only the quantized (perturbed) fields are treated dynamically. This imposes the restriction that the variational derivatives of the Lagrangian with respect to the background (classical) fields and metric vanish. Since these derivatives are in fact coefficient functions (in front of h) in the first-order perturbed gravitational Lagrangian L_1 and the performed expansion (perturbation) is around the fixed background metric, the assumption is equivalent to vanishing of L_1. As will be seen further, such an assumption will no longer hold in this paper, bacause it will be proved that L_1 enters also in the expression for the second order perturbed gravitational Lagrangian L_2 and therefore it has to be taken into account.

It is obvious also from expression (3) that the graviton propagator will no longer have the simple form (in momentum space):

$$D_{\alpha\beta,\lambda\mu}(p)=\frac{1}{2p^2}\left(\delta_{\alpha\lambda}\delta_{\beta\mu}+\delta_{\alpha\mu}\delta_{\beta\lambda}-\delta_{\alpha\beta}\delta_{\lambda\mu}\right) \tag{4}$$

derived in [8], but will have a more complex structure. Indeed, the need to investigate the effect of the gravitational perturbations and of the graviton creation rate in the in flationary universe scenario raised the important problem to find the graviton propagator (derived from second-order perturbative theory) in some real cosmological spacetimes. A number of publications deals with the problem of finding the graviton propagator in homogeneous and isotropic spacetimes [14], in maximally symmetric spacetimes [15] and especially in the de-Sitter spacetime [16-18]. For example, in [19] the graviton propagator in the de-Sitter spacetime has been found to be divergent. However, the graviton propagator has not yet been found for other types of background fields, which are neither highly symmetric, nor homogeneous. The reason is that the quadratic part of the perturbative gravitational Lagrangian has not been yet investigated in details. Perhaps the most serious treatment of this subject remains the paper of Barneby [20], where the formalism of covariant differentiation was successfully applied. However, it is the purpose of this paper to find the most general structure of the second-order Lagrangian and afterwards (in the next papers) the graviton propagator itself. It will be proved in the following paper that some important terms in the second-order Lagrangian and in the Riemann tensor have been omitted in [20], which makes the investigation in this paper incomplete.

Basic assumptions and second-order gravitational Lagrangian decomposition

Our aim in this section will be to decompose the gravitational Lagrangian:

$$L=-\sqrt{g}R\equiv-\sqrt{g}g^{\mu\nu}R_{\mu\nu}\equiv-\sqrt{g}g^{\mu\nu}R^{\alpha}_{\mu\alpha\nu} \tag{5}$$

and to single out those parts of the Lagrangian, corresponding to the background field L_0, to the first-order perturbation of the gravitational field - L_1 and to the second-order perturbation - L_2. Note that in (5) $R^{\alpha}_{\mu\alpha\nu}$ is the Riemann tensor and the geometry of space-time is described by the metric tensor $g_{\mu\nu}$, which is split into a background part $g^{(0)}_{\mu\nu}$ and a fluctuating part $h_{\mu\nu}$:

(6) $$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}.$$

It should be emphasized that (6) is not an expansion around a background geometry, but represents a fluctuating gravitational field, superimposed on a curved background geometry. This fact has a profound cosmological implication, because the fluctuating field may be assumed to be space and time dependent, but the background field may be only time-dependent. It may describe a real cosmological spacetime-a de-Sitter spacetime for example. In order to perform the decomposition (5) according to (6), an inverse metric $g^{\mu\nu}$ to second-order is defined as follows:

(7) $$g^{\mu\beta} = g^{(0)\mu\beta} - h^{\mu\beta} + h^{\mu}_{\alpha} h^{\alpha\beta}.$$

The above formulae should satisfy the relation:

(8) $$g_{\mu\nu} g^{\mu\beta} = \delta^{\beta}_{\nu}$$

to second-order in $h_{\mu\nu}$, unlike the definition in [20], where the inverse metric is defined only to first-order. It is also assumed that an inverse background metric $g^{(0)\mu\nu}$ exists and all indices are raised and lowered with the background metric $g^{(0)}_{\mu\nu}$ and $g^{(0)\mu\nu}$. Since the Riemann tensor $R^{\beta}_{\mu\alpha\nu}$ is given by the well-known formulae:

(9) $$R^{\beta}_{\mu\alpha\nu} \equiv \partial_{\alpha}\Gamma^{\beta}_{\mu\nu} - \partial_{\nu}\Gamma^{\beta}_{\mu\alpha} + \Gamma^{\rho}_{\mu\alpha}\Gamma^{\beta}_{\rho\nu} - \Gamma^{\rho}_{\mu\nu}\Gamma^{\beta}_{\rho\alpha}$$

the Levi-Civita (symmetric) affine connection $\Gamma^{\alpha}_{\mu\nu}$ (with a minus sign):

(10) $$\Gamma^{\alpha}_{\mu\nu} \equiv -\tfrac{1}{2} g^{\alpha s}\left(\partial_{\mu} g_{\nu s} + \partial_{\nu} g_{\mu s} - \partial_{s} g_{\mu\nu}\right).$$

has to be decomposed first. The assumption that formulae (10) is valid for the total affine connection $\Gamma^{\alpha}_{\mu\nu}$ is equivalent to the assumption that the total metric is a Riemannian one, i.e.

(11) $$g_{\mu\nu;\alpha} \equiv \partial_{\alpha} g_{\mu\nu} + \Gamma^{r}_{\mu\alpha} g_{r\nu} + \Gamma^{r}_{\alpha\nu} g_{\mu\nu} \equiv 0.$$

In other words, the covariant derivative (denoted by the symbol ;) of the tensor field $g_{\mu\nu}$ with respect to the total affine connection $\Gamma^{\alpha}_{\mu\nu}$ is zero. In fact, it can easily be proved that formulaes (10) and (11) are equivalent and follow from one another. Of course, an assumption can be made that the background metric is also a Remannian one:

$$g^{(0)}_{\mu\nu|\alpha} = \partial_\alpha g^{(0)}_{\mu\nu} + \Gamma^{(0)r}_{\mu\alpha} g^{(0)}_{r\nu} + \Gamma^{(0)r}_{\alpha\nu} g^{(0)}_{\mu r} \equiv 0 \tag{12}$$

where the symbol | denotes a covariant derivative with respect to the background affine connection $\Gamma^{(0)\alpha}_{\mu\nu}$. Since the background metric is a real cosmological metric and all of them are Riemannian, this is a natural assumption. As will be shown in the following papers, the assumptions (11) and (12) will have profound consequences for this theory. By using (10) and (6) the total Levi - Civita connection can be decomposed into:

$$\Gamma^{\alpha}_{\mu\nu} = \Gamma^{(0)\alpha}_{\mu\nu} + H^{(1)\alpha}_{\mu\nu} + H^{(2)\alpha}_{\mu\nu} \tag{13}$$

where $\Gamma^{(0)\alpha}_{\mu\nu}$ is the usual background affine connection and $H^{(1)\alpha}_{\mu\nu}$ and $H^{(2)\alpha}_{\mu\nu}$ are the first and the second fluctuation connections respectively, given by the expressions:

$$H^{(1)\alpha}_{\mu\nu} \equiv \tfrac{1}{2} h^{\alpha s}\left(\partial_\mu g^{(0)}_{\nu s} + \partial_\nu g^{(0)}_{\mu s} - \partial_s g^{(0)}_{\mu\nu}\right) - \tfrac{1}{2} g^{(0)\alpha s}\left(\partial_\mu h_{\nu s} + \partial_\nu h_{\mu s} - \partial_s h_{\mu\nu}\right), \tag{14}$$

$$H^{(2)\alpha}_{\mu\nu} \equiv \tfrac{1}{2} h^{\alpha s}\left(\partial_\mu h_{\nu s} + \partial_\nu h_{\mu s} - \partial_s h_{\mu\nu}\right) + D^{(2)\alpha}_{\mu\nu} \tag{15}$$

where

$$D^{(2)}_{\mu\nu} \equiv -\tfrac{1}{2} h^{\alpha}_{\gamma} h^{\gamma s}(\partial_\mu g^{(0)}_{\nu s} + \partial_\nu g^{(0)}_{\mu s} - \partial_s g^{(0)}_{\mu\nu}) \equiv h_{s\gamma} h^{s\gamma} \Gamma^{(0)\alpha}_{\mu\nu}. \tag{16}$$

The last formulae is in fact the contribution from the modified (with the term $h^{\mu}_{\alpha} h^{\alpha\beta}$ inverse metric. In (16) the expression:

$$\partial_\mu g^{(0)}_{\nu s} + \partial_\nu g^{(0)}_{\mu s} - \partial_s g^{(0)}_{\mu\nu} \equiv -2 g^{(0)}_{s\delta} \Gamma^{(0)\delta}_{\mu\nu} \tag{17}$$

has been used, which can be obtained from (10) by multiplying both sides with $g^{(0)}_{s\delta}$. From the affine connection formulae decomposition (13) and the Riemann tensor (9) it can be obtained that:

$$R^{\beta}_{\mu\alpha\nu} \equiv R^{(0)\beta}_{\mu\alpha\nu} + R^{(1)\beta}_{\mu\alpha\nu} + R^{(2)\beta}_{\mu\alpha\nu} \tag{18}$$

where $R^{(0)\beta}_{\mu\alpha\nu}$ is the background Riemann tensor and $R^{(1)\beta}_{\mu\alpha\nu}$ and $R^{(2)\beta}_{\mu\alpha\nu}$ are the first and the second fluctuation Riemann tensors respectively:

$$R^{(1)\beta}_{\mu\alpha\nu} \equiv \partial_\alpha H^{(1)\beta}_{\mu\nu} - \partial_\nu H^{(1)\beta}_{\mu\alpha} + H^{(1)\rho}_{\mu\alpha} \Gamma^{(0)\beta}_{\rho\nu} + H^{(1)\beta}_{\rho\nu} \Gamma^{(0)\rho}_{\mu\alpha} - H^{(1)\beta}_{\rho\alpha} \Gamma^{(0)\rho}_{\mu\nu} - H^{(1)\rho}_{\mu\nu} \Gamma^{(0)\beta}_{\rho\alpha}, \tag{19}$$

(20)
$$R^{(2)\beta}_{\mu\alpha\nu} \equiv C^{(2)\beta}_{\mu\alpha\nu} + S^{(2)\beta}_{\mu\alpha\nu} .$$

The tensor $C^{(2)\beta}_{\mu\alpha\nu}$ is the same in structure as the tensor $R^{(1)\beta}_{\mu\alpha\nu}$, but with $H^{(2)\beta}_{\mu\nu}$ (formulae 15) instead of $H^{(1)\beta}_{\mu\nu}$ (formulae 14). The same refers also for the tensor $S^{(2)\beta}_{\mu\alpha\nu}$, obtained from $R^{(1)\beta}_{\mu\alpha\nu}$ by replacing all tensors $H^{(1)\beta}_{\mu\nu}$ with $D^{(2)\beta}_{\mu\nu}$. Using (19) and (16), the tensor $S^{(2)\beta}_{\mu\alpha\nu}$ can be written as:

(21)
$$S^{(2)\beta}_{\mu\alpha\nu} \equiv F^{(2)\beta}_{\mu\alpha\nu} + h^{st} h_{st}\left(\Gamma^{(0)\beta}_{\mu\nu,\alpha} - \Gamma^{(0)\beta}_{\mu\alpha,\nu}\right) + 2\left(\Gamma^{(0)\rho}_{\mu\alpha}\Gamma^{(0)\beta}_{\rho\nu} - \Gamma^{(0)\rho}_{\mu\nu}\Gamma^{(0)\beta}_{\rho\alpha}\right)$$

where $F^{(2)\beta}_{\mu\alpha\nu}$ is, as will be proved later, a tensor quantity:

(22)
$$F^{(2)\beta}_{\mu\alpha\nu} \equiv \left(h_{st}h^{st}\right)_{,\alpha}\Gamma^{(0)\beta}_{\mu\nu} - \left(h_{st}h^{st}\right)_{,\nu}\Gamma^{(0)\beta}_{\mu\alpha} .$$

In (21) and (22) the usual symbol for a partial derivative "∂" has been replaced by a comma ",". Finally, in order to perform the gravitational Lagrangian decomposition, we need also the expression for the decomposition of $\sqrt{g}$:

(23)
$$\sqrt{g} = \sqrt{g}^{(0)}\left(1 + \tfrac{1}{2}h - \tfrac{1}{4}h^{\delta}_{\gamma}h^{\gamma}_{\delta} + \tfrac{1}{8}h^{2}\right)$$

where $h \equiv h^{\alpha}_{\alpha}$. Substituting all expressions (7), (18) - (22) into (5) and representing the Lagrangian as:

(24)
$$L \equiv -L_0 - L_1 - L_2$$

we obtain that

(25)
$$L_0 \equiv \sqrt{g^{(0)}}\, g^{(0)\mu\nu} R^{(0)\alpha}_{\mu\alpha\nu} ,$$

(26)
$$L_1 \equiv \tfrac{1}{2} h L_0 + \sqrt{g^{(0)}}\left(g^{(0)\mu\nu} R^{(1)\alpha}_{\mu\alpha\nu} - h^{\mu\nu} R^{(0)\alpha}_{\mu\alpha\nu}\right),$$

(27)
$$L_2 \equiv \tfrac{1}{4}\left(h^2 - h^{\delta}_{\gamma}h^{\gamma}_{\delta}\right)L_0 + \tfrac{1}{2}hL_1 + \sqrt{g^{(0)}}\left(h^{\mu}_{\delta}h^{\delta\nu}R^{(0)\alpha}_{\mu\alpha\nu} - h^{\mu\nu}R^{(1)\alpha}_{\mu\alpha\nu} + g^{(0)\mu\nu}R^{(2)\alpha}_{\mu\alpha\nu}\right).$$

Discussion

In this paper we have decomposed to second order the total Christoffel connection form $\Gamma^{\alpha}_{\mu\nu}$ of the Riemann metric $g_{\mu\nu}$ (formulaes 13 - 16) and also the total gravitational Lagrangian (formulaes 24 - 27). Note that the second-order Lagrangian (27) is expressed through the preceeding lower-order Lagrangian L_0 (25) and L_1 (26). Also, the Lagrangian L_1 is expressed through L_0.

In paper II it will be proved that the fluctuating Levi - Civita connection $H^{(1)\alpha}_{\mu\nu}$ (14) is a tensor quantity, while the connection $H^{(2)\alpha}_{\mu\nu}$ (15 - 16) is not a tensor. However, it will turn out to be possible to single out from $H^{(2)\alpha}_{\mu\nu}$ a tensor and a non-tensor part and to express the tensor part through the connection $H^{(1)\alpha}_{\mu\nu}$.

This is an important fact since perturbative quantum gravity deals exclusively with tensor quantities.

Acknowledgements. The author is grateful to Dr S. Manoff from the Theoretical Physics Department at the Institute for Nuclear Research and Nuclear Energy in the Bulgarian Academy of Sciences for the numerous and helpful discussions, advises and critical remarks.

References

1. Dowker, H.F., R. Laflamme.- Nucl. Phys. B, 366, 1991, 209 - 232.
2. Tsamis, N.C., R. P. Woodard. Relaxing the Cosmological Constant. Institute for Fundamental Theory Preprint UFIFT - HEP - 92-23, Crete-92-15, October 1992, University of Florida.
3. De Witt, S. Bryce. Dynamical Theory of Groups and Fields. New York, Gordon and Breach, 1965.
4. De Witt, S. Bryce. - Phys. Rev. , 162, 1967, Nr. 5, 1195 - 1239.
5. De Witt, S. Bryce. - Phys. Rev. , 162, 1967, Nr. 5, 1239 - 1256 .
6. Isham, C.J., R. Penrose, D. W. Sciama (eds.). Quantum Gravity 2. Oxford, Claredon Press, 1981.
7. Stelle, K. - Phys. Rev. D, 16, 1977, Nr. 4, 953 - 969.
8. Capper, D.M., G. Liebrandt, M. Ramon - Medrano. - Phys. Rev. D, 8, 1973, N. 12, 4320 - 4331.
9. Capper, D. M., M. Ramon - Medrano. - Phys. Rev. D, 9, 1974, Nr. 6, 1641 - 1647.
10. T' Hooft, G., M. Veltman. - Ann. Inst. Henry Poincare, 20, 1974, Nr. 1, 69 - 94.
11. Alvarez, Enrique. - Rev. Mod. Phys., 61, 1989, Nr. 3, 561 - 603.
12. Deser, S., P. van Nieuwenhuizen. - Phys. Rev. D, 10, 1974, Nr. 2, 401 - 410.
13. Landau, L.D., E. M. Lifschitz. A Course of Theoretical Physics. Vol. II. Field Theory, Moscow, Nauka Publishing House, 1988.
14. Allen, Bruce. - Nucl. Phys. B, 287, 1987, 743 - 756.
15. Turyn, Michael. - J. Math. Phys., 31, 1990, Nr. 3, 669 - 679.
16. Allen, Bruce. - Phys. Rev. D, 34, 1986, Nr. 12, 3670 - 3675.
17. Allen, Bruce, Michael Turyn. - Nucl. Phys. B, 29, 1987, 813 - 852.
18. Antoniadis, I., J. Iliopoulos, T. N. Tomaras. - Phys. Rev. Lett., 56, 1986, Nr. 13, 1319 - 1322.

19. Antoniadis, I., E. Mottola. - J. Math. Phys. 32, 1991, Nr. 4, 1037 - 1044.
20. Barneby, T. A. - Phys. Rev. D, 10, 1974, Nr. 6, 1741 - 1748.

Received 15. XI. 1995

Върху теорията на пертурбативната гравитация и квантовата гравитация на изкривен фон. I. Разлагане на гравитационния Лагражиан до втори порядък

Богдан Димитров

(Резюме)

В тази първа работа от серия от няколко работи стандартният гравитационен Лагражиан е разложен до втори порядък по отношение на пертурбации на гравитационното поле. Направено е разлагане до втори порядък също и на свързаността, и на тензора на Риман.

В работата е установено, че Лагражианът от втори порядък се изразява чрез Лагражиана от първи порядък и този на фоновото поле.